

The q -analog of higher order Hochschild homology and the Lie derivative

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Abstract

Let A be a commutative algebra over \mathbb{C} . Given a pointed simplicial finite set Y and $q \in \mathbb{C}$ a primitive N -th root of unity, we define the q -Hochschild homology groups $\{ {}_qHH_n^Y(A) \}_{n \geq 0}$ of A of order Y . When D is a derivation on A , we construct the corresponding Lie derivative on the groups $\{ {}_qHH_n^Y(A) \}_{n \geq 0}$. We also define the Lie derivative on $\{ {}_qHH_n^Y(A) \}_{n \geq 0}$ for a higher derivation $\{ D_n \}_{n \geq 0}$ on A . Finally, we describe the morphisms induced on the bivariate q -Hochschild cohomology groups $\{ {}_qHH_Y^n(A, A) \}_{n \in \mathbb{Z}}$ of order Y by a derivation D on A .

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1 Introduction

Let A be a commutative algebra over \mathbb{C} . Then, it is well known (see, for instance, [7, § 4.1]) that a derivation $D : A \longrightarrow A$ induces morphisms

$$L_D^n : HH_n(A) \longrightarrow HH_n(A) \quad \forall n \geq 0 \quad (1.1)$$

on the Hochschild homology groups of the algebra A . The morphisms L_D^n , $n \geq 0$ play the role of the Lie derivative in noncommutative geometry. For more on these morphisms and for general properties of Hochschild homology, we refer the reader to [7]. Further, for any pointed simplicial finite set Y , Pirashvili [10] has introduced the Hochschild homology groups $\{ HH_n^Y(A) \}_{n \geq 0}$ of A of order Y (see also Loday [6]). In particular, when $Y = S^1$ is the simplicial circle, the groups $\{ HH_n^{S^1}(A) \}_{n \geq 0}$ reduce to the usual Hochschild homology groups of the algebra A . Let $q \in \mathbb{C}$ be a primitive N -th root of unity. The purpose of this paper is to introduce the q -analogues $\{ {}_qHH_n^Y(A) \}_{n \geq 0}$ of these higher order Hochschild homology groups and study the morphisms induced on them by derivations on A .

More precisely, let Γ denote the category whose objects are the finite sets $[n] = \{0, 1, 2, \dots, n\}$, $n \geq 0$ with basepoint $0 \in [n]$. Then, given the algebra A , we can define a functor $\mathcal{L}(A)$ from Γ to the category $Vect$ of complex vector spaces that takes $[n]$ to $A \otimes A^{\otimes n}$ (see Section 2 for details). Then, we can

prolong $\mathcal{L}(A)$ by means of colimits to a functor $\mathcal{L}(A) : Fin_* \rightarrow Vect$ from the category Fin_* of all finite sets with basepoint. Given a pointed simplicial finite set Y , i.e., a functor $Y : \Delta^{op} \rightarrow Fin_*$ (Δ^{op} being the simplex category), we now have a simplicial vector space

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect \quad (1.2)$$

Let $d_i^j : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1}$, $0 \leq j \leq i$, $i \geq 0$ be the face maps of the simplicial vector space $\mathcal{L}^Y(A)$. We then construct the “ q -Hochschild differentials”:

$${}_q b_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1} \quad {}_q b_i := \sum_{j=0}^i q^j d_i^j \quad (1.3)$$

Since $q \in \mathbb{C}$ is a primitive N -th root of unity, it follows that ${}_q b^N = 0$, i.e., $(\mathcal{L}^Y(A), {}_q b)$ is an N -complex in the sense of Kapranov [5]. We now define the q -Hochschild homology groups $\{{}_q HH_n^Y(A)\}_{n \geq 0}$ of A of order Y to be the homology objects of the N -complex $(\mathcal{L}^Y(A), {}_q b)$ (see Definitions 2.1 and 2.2). When $q = -1$ (and hence $N = 2$), ${}_q b$ reduces to the usual differential on the chain complex associated to the simplicial vector space $\mathcal{L}^Y(A)$ and we have ${}_{(-1)} HH_n^Y(A) = HH_n^Y(A)$, $\forall n \geq 0$. Then, the main result of Section 2 is as follows.

Theorem 1.1. *Let $D : A \rightarrow A$ be a derivation on A . Then, for each $n \geq 0$, the derivation D induces a morphism $L_D^{Y,n} : {}_q HH_n^Y(A) \rightarrow {}_q HH_n^Y(A)$ of q -Hochschild homology groups of order Y . Additionally, if $\mathcal{H} = \mathcal{U}(Der(A))$ is the universal enveloping algebra of the Lie algebra $Der(A)$ of derivations on A , each ${}_q HH_n^Y(A)$ is a left \mathcal{H} -module, i.e., for any element $h \in \mathcal{H}$, there exist morphisms $L_h^{Y,n} : {}_q HH_n^Y(A) \rightarrow {}_q HH_n^Y(A)$ of q -Hochschild homology groups of order Y .*

Thereafter, we consider a higher derivation $D = \{D_n\}_{n \geq 0}$ on A . We recall that a higher (or Hasse-Schmidt) derivation $D = \{D_n\}_{n \geq 0}$ on A is a sequence of linear maps $D_n : A \rightarrow A$ satisfying the following relation (see, for example, [8]):

$$D_n(a \cdot a') = \sum_{i=0}^n D_i(a) \cdot D_{n-i}(a') \quad \forall a, a' \in A, n \geq 0 \quad (1.4)$$

In this paper, we restrict ourselves to normalized higher derivations, i.e., higher derivations $D = \{D_n\}_{n \geq 0}$ such that $D_0 = 1$. Then, in Section 3, we construct the Lie derivative on the q -Hochschild homology groups of order Y corresponding to a higher derivation $D = \{D_n\}_{n \geq 0}$.

Theorem 1.2. *Let $D = \{D_n\}_{n \geq 0}$ be a normalized higher derivation on A . Then, for each $k \geq 0$, we have an induced morphism $L_D^{Y,k} : {}_q HH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_q HH_n^Y(A) \rightarrow {}_q HH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_q HH_n^Y(A)$ on the q -Hochschild homology groups of A of order Y .*

Further, in [9], Mirzavaziri has provided a characterization of normalized higher derivations on algebras over \mathbb{C} from which it follows that if $D = \{D_k\}_{k \geq 0}$ is a higher derivation on A , each D_k is an element of the Hopf algebra $\mathcal{H} = \mathcal{U}(Der(A))$ (see (3.7) for details). It follows therefore from Theorem 1.1 that for each k , the element $D_k \in \mathcal{H}$ induces a morphism $L_{D_k}^Y : {}_q HH_*^Y(A) \rightarrow {}_q HH_*^Y(A)$. Then, in Section 3, we prove the following result.

Theorem 1.3. *Let $D = \{D_k\}_{k \geq 0}$ be a normalized higher derivation on A . Then, for each $k \geq 1$, we have $L_{D_k}^Y = L_D^{Y,k}$ as an endomorphism of ${}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A)$.*

In Section 4, we start by defining bivariant q -Hochschild cohomology groups $\{{}_qHH_Y^n(A, A)\}_{n \in \mathbb{Z}}$ of order Y . For this we consider the modules $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, $n \in \mathbb{Z}$ where an element $f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$ is given by a family of morphisms $f = \{f_i : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i+n}\}_{i \in \mathbb{Z}}$. Further, we define a differential ${}_q\partial_n : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n \rightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{n-1}$ (see Definition 4.1). Then, $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$ is an N -complex and we let ${}_qHH_Y^n(A, A)$ be the $(-n)$ -th homology object of $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$. We end with the following result.

Theorem 1.4. *Let $D : A \rightarrow A$ be a derivation on A . Then, for each $n \in \mathbb{Z}$, the derivation D induces a morphism $\underline{L}_D^{Y,n} : {}_qHH_Y^n(A, A) \rightarrow {}_qHH_Y^n(A, A)$ on the bivariant q -Hochschild cohomology groups of order Y .*

2 Lie Derivative on higher order q -Hochschild homology

Let $Vect$ denote the category of vector spaces over \mathbb{C} . Let A be a commutative \mathbb{C} -algebra. We recall here the definition of higher order Hochschild homology groups of a commutative algebra A as introduced by Pirashvili [10] (see also Loday [6]). Let Γ denote the category whose objects are the pointed sets $[n] = \{0, 1, 2, \dots, n\}$ with $0 \in [n]$ as base point for each $n \geq 0$. Then, a morphism $\phi : [m] \rightarrow [n]$ in Γ is a map $\phi : \{0, 1, 2, \dots, m\} \rightarrow \{0, 1, 2, \dots, n\}$ of sets such that $\phi(0) = 0$. We now define a functor:

$$\mathcal{L}(A) : \Gamma \rightarrow Vect \quad [n] \mapsto A \otimes A^{\otimes n} \quad (2.1)$$

Given a morphism $\phi : [m] \rightarrow [n]$ in Γ , we have an induced map in $Vect$:

$$\begin{aligned} \mathcal{L}(A)(\phi) : A \otimes A^{\otimes m} &\rightarrow A \otimes A^{\otimes n} \\ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes \dots \otimes a_m) &= (b_0 \otimes b_1 \otimes \dots \otimes b_n) \quad b_j := \prod_{\phi(i)=j} a_i \end{aligned} \quad (2.2)$$

We now consider the category Fin_* of finite pointed sets. There is a natural inclusion $\Gamma \hookrightarrow Fin_*$ of categories. Then, $\mathcal{L}(A) : \Gamma \rightarrow Vect$ can be extended to a functor $\mathcal{L}(A) : Fin_* \rightarrow Vect$ by setting:

$$\mathcal{L}(A) : Fin_* \rightarrow Vect \quad T \mapsto \operatorname{colim}_{\Gamma \ni T' \rightarrow T} \mathcal{L}(A)(T') \quad (2.3)$$

where the colimit in (2.3) is taken over all morphisms $T' \rightarrow T$ in Fin_* such that $T' \in \Gamma$. Let Δ be the simplex category, i.e., the category whose objects are sets $[n] = \{0, 1, 2, \dots, n\}$, $n \geq 0$ and whose morphisms are order preserving maps. Then, given a pointed simplicial finite set Y corresponding to a functor $Y : \Delta^{op} \rightarrow Fin_*$, we have a simplicial vector space $\mathcal{L}^Y(A)$ determined by the composition of functors:

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect \quad (2.4)$$

For any $n \geq 0$, let $HH_n^Y(A)$ denote the n -th homology group of the chain complex associated to the simplicial vector space $\mathcal{L}^Y(A)$. Following Pirashvili [10], when $Y = S^p$ (S^p being the sphere of dimension $p \geq 1$), we say that the homology groups $\{HH_n^{S^p}(A)\}_{n \geq 0}$ are the Hochschild homology groups of A of order p . When $p = 1$, i.e., $Y = S^1$ is the simplicial circle, the Hochschild homology groups $\{HH_n^{S^1}(A)\}_{n \geq 0}$ are identical to the usual Hochschild homology groups of A .

Our objective is to introduce a q -analog of the groups $HH_*^Y(A)$, where $q \in \mathbb{C}$ is a primitive N -th root of unity. For this, we consider the face maps $d_n^i : \mathcal{L}^Y(A)_n \rightarrow \mathcal{L}^Y(A)_{n-1}$, $0 \leq i \leq n$, $n \geq 0$, of the simplicial vector space $\mathcal{L}^Y(A)$ defined in (2.4). We set:

$${}_q b_n : \mathcal{L}^Y(A)_n \rightarrow \mathcal{L}^Y(A)_{n-1} \quad {}_q b_n := \sum_{i=0}^n q^i d_n^i \quad (2.5)$$

For the sake of convenience, we will often write ${}_q b_n$ simply as ${}_q b$. Then, it is well known that the morphism ${}_q b$ satisfies ${}_q b^N = 0$ (this is true in general for any simplicial vector space; see, for instance, Kapranov [5, Proposition 0.2]). In particular, if $q = -1$, i.e., $N = 2$, we have $(-1)b^2 = 0$ and $(-1)b$ is the standard differential on the chain complex corresponding to the simplicial vector space $\mathcal{L}^Y(A)$. In general, the pair $(\mathcal{L}^Y(A), {}_q b)$, i.e., the simplicial vector space $\mathcal{L}^Y(A)$ equipped with the morphism ${}_q b$ is an “ N -complex” in the sense defined below.

Definition 2.1. (see [3, § 2] and [5, Definition 0.1]) Let \mathcal{A} be an abelian category and $N \geq 2$ a positive integer. An N -complex in \mathcal{A} is a sequence of objects and morphisms of \mathcal{A}

$$C_* = \{\dots \rightarrow C_1 \xrightarrow{b_1} C_0 \xrightarrow{b_0} C_{-1} \rightarrow \dots\} \quad (2.6)$$

such that the composition of any N consecutive morphisms in (2.6) is 0. For any $n \in \mathbb{Z}$, the homology object $H_{\{n\}}(C_*, b)$ of the N -complex (C_*, b) is defined as:

$$H_{\{n\}}(C_*, b) := \bigoplus_{i=1}^{N-1} H_{\{i,n\}}(C_*, b) \quad H_{\{i,n\}}(C_*, b) := \frac{\text{Ker}(b^i : C_n \rightarrow C_{n-i})}{\text{Im}(b^{N-i} : C_{N-i+n} \rightarrow C_n)} \quad (2.7)$$

Definition 2.2. Let A be a commutative algebra over \mathbb{C} and let Y be a pointed simplicial finite set. Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Then, the q -Hochschild homology groups ${}_q HH_n^Y(A)$, $n \geq 0$ of A of order Y are defined to be the homology objects of the N -complex $(\mathcal{L}^Y(A), {}_q b)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$; in other words, we define:

$${}_q HH_n^Y(A) := H_{\{n\}}(\mathcal{L}^Y(A), {}_q b) \quad (2.8)$$

As with the ordinary Hochschild homology of an algebra (see, for instance, [7, § 4.1]), given a derivation $D : A \rightarrow A$, we want to construct the Lie derivative $L_D^Y : HH_*^Y(A) \rightarrow HH_*^Y(A)$ on the Hochschild homology of order Y . For this, we start with the following lemma.

Lemma 2.3. *Let A be a commutative \mathbb{C} -algebra and let $D : A \longrightarrow A$ be a derivation on A . Then, the derivation D induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$.*

Proof. We first consider the functor $\mathcal{L}(A)$ restricted to the subcategory Γ of Fin_* , defined as in (2.1) and (2.2):

$$\mathcal{L}(A) : \Gamma \longrightarrow Vect \quad [n] \mapsto A \otimes A^{\otimes n} \quad (2.9)$$

Given the derivation D on A , we define morphisms (for all $n \geq 0$):

$$L_D([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) \quad (a_0 \otimes a_1 \otimes \dots \otimes a_n) \mapsto \sum_{i=0}^n (a_0 \otimes a_1 \otimes \dots \otimes D(a_i) \otimes \dots \otimes a_n) \quad (2.10)$$

Further, for any morphism $\phi : [m] \longrightarrow [n]$ in Γ , we have, for any $(a_0 \otimes a_1 \otimes \dots \otimes a_m) \in A \otimes A^{\otimes m}$:

$$\begin{aligned} L_D([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes \dots \otimes a_m) &= L_D([n]) \left(\bigotimes_{j=0}^n \prod_{\phi(i)=j} a_i \right) \\ &= \sum_{k=0}^n \left(\bigotimes_{j=0}^{k-1} \prod_{\phi(i)=j} a_i \right) \otimes D \left(\prod_{\phi(i)=k} a_i \right) \otimes \left(\bigotimes_{j=k+1}^n \prod_{\phi(i)=j} a_i \right) \\ &= \sum_{k=0}^n \sum_{i \in \phi^{-1}(k)} \mathcal{L}(A)(\phi)(a_0 \otimes \dots \otimes D(a_i) \otimes \dots \otimes a_m) \\ &= \sum_{i=0}^m \mathcal{L}(A)(\phi)(a_0 \otimes \dots \otimes D(a_i) \otimes \dots \otimes a_m) \\ &= \mathcal{L}(A)(\phi) \circ L_D([m])(a_0 \otimes a_1 \otimes \dots \otimes a_m) \end{aligned}$$

It follows that the derivation D induces an endomorphism L_D of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. More generally, for any object $T \in Fin_*$ and a morphism $T' \longrightarrow T$ in Fin_* such that $T' \in \Gamma$, we have a morphism $L_D(T') : \mathcal{L}(A)(T') \longrightarrow \mathcal{L}(A)(T')$ as defined in (2.10). By definition, we know that $\mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \longrightarrow T} \mathcal{L}(A)(T')$ and hence we have an induced morphism

$$L_D(T) : \mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \longrightarrow T} \mathcal{L}(A)(T') \longrightarrow \mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \longrightarrow T} \mathcal{L}(A)(T') \quad (2.11)$$

From (2.11) it follows that the derivation D induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. This proves the claim. \square

Proposition 2.4. *Let A be a commutative \mathbb{C} -algebra and let $D : A \longrightarrow A$ be a derivation on A . Let Y be a pointed simplicial finite set. Then, for each $n \geq 0$, the derivation D induces a morphism $L_D^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ of q -Hochschild homology groups of order Y , where $q \in \mathbb{C}$ is a primitive N -th root of unity.*

Proof. From Lemma 2.3, we know that the derivation D induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Given the pointed simplicial finite set Y , the endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of functors induces an endomorphism of the functor

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect \quad (2.12)$$

From (2.12), it follows that we have an endomorphism $L_D^Y : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the simplicial vector space $\mathcal{L}^Y(A)$. Hence, we have induced morphisms $L_D^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ on the homology objects of the N -complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5). \square

We now let $Der(A)$ denote the vector space of all derivations on the commutative \mathbb{C} -algebra A . Then, $Der(A)$ is a Lie algebra, endowed with the Lie bracket $[D, D'] := D \circ D' - D' \circ D$, $\forall D, D' \in Der(A)$. Let $\mathcal{H} := \mathcal{U}(Der(A))$ denote the universal enveloping algebra of $Der(A)$. We will now show that for any pointed simplicial finite set Y , the operators $L_D^{Y,n}$, $D \in Der(A)$ on the q -Hochschild homology group of A of order Y make ${}_qHH_n^Y(A)$ into a module over the Hopf algebra $\mathcal{H} = \mathcal{U}(Der(A))$.

Lemma 2.5. *Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let A be a commutative \mathbb{C} -algebra and let $D, D' \in Der(A)$ be derivations on A . Let Y be a pointed simplicial finite set. Then, for each $n \geq 0$, the operators $L_D^{Y,n}, L_{D'}^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ satisfy $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D, D']}^{Y,n}$.*

Proof. For $D, D' \in Der(A)$, we consider the respective endomorphisms $L_D, L_{D'}$ of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. By definition, for any object $[n] \in \Gamma$, we have morphisms:

$$\begin{aligned} L_D([n]) : \mathcal{L}(A)([n]) &\longrightarrow \mathcal{L}(A)([n]) & (a_0 \otimes \dots \otimes a_n) &\mapsto \sum_{i=0}^n (a_0 \otimes \dots \otimes D(a_i) \otimes \dots \otimes a_n) \\ L_{D'}([n]) : \mathcal{L}(A)([n]) &\longrightarrow \mathcal{L}(A)([n]) & (a_0 \otimes \dots \otimes a_n) &\mapsto \sum_{i=0}^n (a_0 \otimes \dots \otimes D'(a_i) \otimes \dots \otimes a_n) \end{aligned} \quad (2.13)$$

From (2.13), it may be verified easily that we have

$$(L_D \circ L_{D'} - L_{D'} \circ L_D)([n]) = L_{[D, D']}([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) \quad \forall n \geq 0 \quad (2.14)$$

and it follows that $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{[D, D']}$ as endomorphisms of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. More generally, for any object $T \in Fin_*$, we have $\mathcal{L}(A)(T) = \text{colim}_{\Gamma \ni T' \rightarrow T} \mathcal{L}(A)(T')$ and hence $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{[D, D']}$ as endomorphisms of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Finally, considering the composition of $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ with the functor $Y : \Delta^{op} \longrightarrow Fin_*$ corresponding to the pointed simplicial finite set Y , it follows that $L_D^Y \circ L_{D'}^Y - L_{D'}^Y \circ L_D^Y = L_{[D, D']}^Y$ as endomorphisms of the functor $\mathcal{L}^Y(A) : \Delta^{op} \longrightarrow Vect$. Hence, we have $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D, D']}^{Y,n}$ on the homology objects ${}_qHH_n^Y(A)$, $n \geq 0$ of the N -complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A) : \Delta^{op} \longrightarrow Vect$ as in (2.5). \square

Proposition 2.6. *Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let A be a commutative algebra over \mathbb{C} and let $Der(A)$ denote the Lie algebra of derivations on A . Let $\mathcal{H} = \mathcal{U}(Der(A))$ denote the universal enveloping algebra of $Der(A)$. Then, for any pointed simplicial finite set Y and any $n \geq 0$, the q -Hochschild homology group ${}_qHH_n^Y(A)$ of order Y is a left module over the Hopf algebra \mathcal{H} .*

Proof. From Lemma 2.5, it follows that $Der(A)$ has a Lie algebra action on each ${}_qHH_n^Y(A)$, i.e., $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D, D']}^{Y,n}$ for any $D, D' \in Der(A)$. Since \mathcal{H} is the universal enveloping algebra of $Der(A)$, it follows that this Lie algebra action of $Der(A)$ on ${}_qHH_n^Y(A)$ makes ${}_qHH_n^Y(A)$ into a left \mathcal{H} -module. \square

3 Higher derivations and the Lie derivative

As before, we work with a commutative algebra A over \mathbb{C} , a pointed simplicial finite set Y and $q \in \mathbb{C}$ a primitive N -th root of unity. In this section, we will describe the Lie derivative on the q -Hochschild homology groups ${}_q HH_*^Y(A)$ corresponding to a higher derivation D on A . Given an ordinary derivation d on A , it is easy to verify that the sequence $\{D_n := d^n/n!\}_{n \geq 0}$ satisfies the following identity:

$$D_n(a \cdot a') = \sum_{i=0}^n D_i(a) \cdot D_{n-i}(a') \quad \forall n \geq 0, a, a' \in A \quad (3.1)$$

More generally, we have the notion of a higher (or Hasse-Schmidt) derivation on A .

Definition 3.1. (see, for instance, [8]) Let A be a commutative algebra over \mathbb{C} . A sequence $D = \{D_n\}_{n \geq 0}$ of \mathbb{C} -linear maps on A is said to be a higher (or Hasse-Schmidt) derivation on A if it satisfies:

$$D_n(a \cdot a') = \sum_{i=0}^n D_i(a) \cdot D_{n-i}(a') \quad \forall n \geq 0, a, a' \in A \quad (3.2)$$

In this paper, we will only work with higher derivations $D = \{D_n\}_{n \geq 0}$ that are normalized, i.e., those higher derivations $D = \{D_n\}_{n \geq 0}$ which satisfy $D_0 = 1$. For a normalized higher derivation $D = \{D_n\}_{n \geq 0}$ it is easy to verify from relation (3.2) that $D_n(1) = 0$ for all $n > 0$. For more on the structure of higher derivations on an algebra, we refer the reader to [9], [11] and [12]. For a higher derivation on A , we have already described in [2] the corresponding Lie derivative on the ordinary Hochschild homology; we are now ready to introduce the action of a higher derivation on the q -Hochschild homology groups of order Y of the algebra A .

Lemma 3.2. Let A be a commutative algebra over \mathbb{C} and let $D = \{D_n\}_{n \geq 0}$ be a (normalized) higher derivation on A . Then, for any given $k \geq 0$, the higher derivation D induces an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$.

Proof. It suffices to prove that for each $k \geq 0$, we have an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ restricted to the subcategory Γ of Fin_* . Given the higher derivation $D = \{D_n\}_{n \geq 0}$ and the integer $k \geq 0$, we define morphisms ($\forall n \geq 0$)

$$L_D^k([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n])$$

$$(a_0 \otimes a_1 \otimes \dots \otimes a_n) \mapsto \sum_{\substack{(p_0, p_1, \dots, p_n) \\ p_0 + p_1 + \dots + p_n = k}} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes \dots \otimes D_{p_n}(a_n)) \quad (3.3)$$

For the sake of convenience, we will often denote a sum as in (3.3) taken over all ordered tuples (p_0, p_1, \dots, p_n) of non-negative integers such that $p_0 + p_1 + \dots + p_n = k$ simply as

$$(a_0 \otimes a_1 \otimes \dots \otimes a_n) \mapsto \sum_{p_0 + p_1 + \dots + p_n = k} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes \dots \otimes D_{p_n}(a_n)) \quad (3.4)$$

Let $\phi : [m] \longrightarrow [n]$ be a morphism in Γ . We let $N(j)$ denote the cardinality of the set $\phi^{-1}(j) \subseteq [m]$ for any $0 \leq j \leq n$. Then, we have, for any $(a_0 \otimes a_1 \otimes \dots \otimes a_m) \in A \otimes A^{\otimes m}$:

$$\begin{aligned}
L_D^k([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes \dots \otimes a_m) &= L_D^k([n]) \left(\bigotimes_{j=0}^n \prod_{\phi(i)=j} a_i \right) \\
&= \sum_{p_0+p_1+\dots+p_n=k} \left(\bigotimes_{j=0}^n D_{p_j} \left(\prod_{\phi(i)=j} a_i \right) \right) \\
&= \sum_{p_0+p_1+\dots+p_n=k} \left(\bigotimes_{j=0}^n \sum_{q_1+\dots+q_{N(j)}=p_j} \prod_{\phi(i)=j} D_{q_i}(a_i) \right) \quad (3.5) \\
&= \sum_{r_0+r_1+\dots+r_m=k} \left(\bigotimes_{j=0}^n \prod_{\phi(i)=j} D_{r_i}(a_i) \right) \\
&= \sum_{r_0+r_1+\dots+r_m=k} \mathcal{L}(A)(\phi) \left(\bigotimes_{i=0}^m D_{r_i}(a_i) \right) \\
&= \mathcal{L}(A)(\phi) \circ L_D^k([m])(a_0 \otimes a_1 \otimes \dots \otimes a_m)
\end{aligned}$$

From (3.5), it follows that for each $k \geq 0$, $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ is an endomorphism of the functor $\mathcal{L}(A)$ restricted to Γ and hence, taking colimits as in the proof of Lemma 2.3, L_D^k induces an endomorphism of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. □

Proposition 3.3. *Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let A be a commutative algebra over \mathbb{C} and let Y be a pointed simplicial finite set. Then, given a higher derivation $D = \{D_n\}_{n \geq 0}$ on A , for each $k \geq 0$, we have an induced morphism:*

$$L_D^{Y,k} : {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A) \longrightarrow {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A) \quad (3.6)$$

on the q -Hochschild homology groups of A of order Y .

Proof. From Lemma 3.2, we know that for any $k \geq 0$, we have an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Composing with the functor $Y : \Delta^{op} \longrightarrow Fin_*$ corresponding to the pointed simplicial finite set Y , we have an induced endomorphism $L_D^{Y,k} : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect$. Accordingly, $L_D^{Y,k}$ induces an endomorphism on the homology objects of the N -complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5). Hence, we have induced morphisms $L_D^{Y,k} : {}_qHH_*^Y(A) \longrightarrow {}_qHH_*^Y(A)$ on the q -Hochschild homology groups of order Y . □

We have already shown in the last section that ${}_qHH_*^Y(A)$ is a left module over the universal enveloping algebra $\mathcal{H} = \mathcal{U}(Der(A))$ of the Lie algebra of derivations on A . Given a higher derivation $D = \{D_k\}_{k \geq 0}$

on a \mathbb{C} -algebra A , Mirzavaziri [9] has shown that the higher derivation D may be expressed as follows: there exists a sequence of ordinary derivations $\{d_n\}_{n \geq 0}$, $d_n \in \text{Der}(A)$ such that:

$$D_k = \sum_{i=1}^k \left(\sum_{\sum_{j=1}^i r_j = k} \left(\prod_{j=1}^i \frac{1}{r_j + \dots + r_i} \right) d_{r_1} \dots d_{r_i} \right) \quad (3.7)$$

From (3.7), it is clear that given a higher derivation $D = \{D_k\}_{k \geq 0}$ on A , each D_k is an element of the Hopf algebra $\mathcal{H} = \mathcal{U}(\text{Der}(A))$. Hence, it follows from Proposition 2.6 that each operator $D_k \in \mathcal{H}$ induces a morphism $L_{D_k}^Y : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ on the q -Hochschild homology groups of order Y . We will now show that the morphisms $L_{D_k}^Y$, $k \geq 1$ are identical to the morphisms $L_D^{Y,k} : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ described in Proposition 3.3.

Proposition 3.4. *Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let A be a commutative algebra over \mathbb{C} and let Y be a pointed simplicial finite set. Let $D = \{D_k\}_{k \geq 0}$ denote a higher derivation on A . For any $k \geq 1$, let $L_{D_k}^Y : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ be the morphism induced by $D_k \in \mathcal{H}$ as in Proposition 2.6 and let $L_D^{Y,k} : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ be the morphism induced by D as in Proposition 3.3. Then, we have $L_{D_k}^Y = L_D^{Y,k} : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$.*

Proof. From the proofs of Lemma 2.3 and Lemma 2.5, it follows that the element $D_k \in \mathcal{H} = \mathcal{U}(\text{Der}(A))$ of the universal enveloping algebra \mathcal{H} defines an endomorphism $L_{D_k} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$. From the proofs of Proposition 2.4 and Proposition 2.6, it is clear that the morphism $L_{D_k}^Y : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ is obtained from the endomorphism $L_{D_k}^Y : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y$ induced by $L_{D_k} : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$.

Similarly, from Lemma 3.2, it follows that the higher derivation D induces an endomorphism $L_D^k : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$. From the proof of Proposition 3.3, it follows that the morphism $L_D^{Y,k} : {}_qHH_*^Y(A) \rightarrow {}_qHH_*^Y(A)$ is obtained from the endomorphism $L_D^{Y,k} : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y$ induced by $L_D^k : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$. Hence, in order to prove the result, we need to show that $L_D^k = L_{D_k}$ as endomorphisms of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$. As before, it suffices to show that $L_D^k = L_{D_k}$ as endomorphisms of the functor $\mathcal{L}(A)$ restricted to the subcategory Γ of Fin_* .

Let $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ denote the coproduct on \mathcal{H} . For any $h \in \mathcal{H}$ and any $n \geq 0$, we write $\Delta^n(h) = \sum h_{(1)} \otimes h_{(2)} \otimes \dots \otimes h_{(n+1)}$. Then, we have an induced endomorphism $L_h : \mathcal{L}(A) \rightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : \text{Fin}_* \rightarrow \text{Vect}$. Further, we note that the equation

$$L_h([n])(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \sum (h_{(1)}(a_0) \otimes h_{(2)}(a_1) \otimes \dots \otimes h_{(n+1)}(a_n)) \quad \forall (a_0 \otimes \dots \otimes a_n) \in \mathcal{L}(A)([n]) \quad (3.8)$$

holds for all $h \in \text{Der}(A) \subseteq \mathcal{H}$ and hence for all $h \in \mathcal{H} = \mathcal{U}(\text{Der}(A))$. From the definition of L_D^k in Lemma 3.2, we now see that in order to show that $L_D^k = L_{D_k}$, it suffices to show that

$$\Delta^n(D_k) = \sum_{\sum_{i=0}^n p_i = k} D_{p_0} \otimes D_{p_1} \otimes \dots \otimes D_{p_n} \quad \forall n \geq 0 \quad (3.9)$$

We will prove (3.9) by induction on k . For any given $n \geq 0$, it is clear that the equation (3.9) holds for $k = 0$ and $k = 1$. We now suppose that it holds for any $0 \leq k \leq K$. From [9, Proposition 2.1], we know that

$$D_{M+1} = \frac{1}{M+1} \sum_{m=0}^M d_{m+1} D_{M-m} \quad \forall M \geq 0 \quad (3.10)$$

where the d_{m+1} are the derivations corresponding to the higher derivation $D = \{D_n\}_{n \geq 0}$ as described in (3.7). From (3.10), it follows that $\Delta^n(D_{K+1}) = \frac{1}{K+1} \sum_{m=0}^K \Delta^n(d_{m+1}) \Delta^n(D_{K-m})$ and hence

$$\Delta^n(D_{K+1}) = \frac{1}{K+1} \sum_{m=0}^K \left(\sum_{j=0}^n d_{m+1}^j \right) \left(\sum_{\sum_{i=0}^n p_i = K-m} D_{p_0} \otimes D_{p_1} \otimes \dots \otimes D_{p_n} \right) \quad (3.11)$$

where d_{m+1}^j denotes the term $1 \otimes 1 \otimes \dots \otimes d_{m+1} \otimes \dots \otimes 1$ (i.e., d_{m+1} at the j -th position) appearing in the expression for $\Delta^n(d_{m+1})$. We now consider ordered tuples $(p'_0, p'_1, \dots, p'_n)$ of non-negative integers such that $p'_0 + p'_1 + \dots + p'_n = K+1$. Then, we can write:

$$\begin{aligned} & \sum_{m=0}^K \left(\sum_{j=0}^n d_{m+1}^j \right) \left(\sum_{\sum_{i=0}^n p_i = K-m} D_{p_0} \otimes D_{p_1} \otimes \dots \otimes D_{p_n} \right) \\ &= \sum_{\sum_{i=0}^n p'_i = K+1} \sum_{j=0, p'_j \geq 1}^n \sum_{m=0}^{p'_j-1} d_{m+1}^j \cdot (D_{p'_0} \otimes \dots \otimes D_{p'_j-m-1} \otimes \dots \otimes D_{p'_n}) \\ &= \sum_{\sum_{i=0}^n p'_i = K+1} \sum_{j=0, p'_j \geq 1}^n \sum_{m=0}^{p'_j-1} (D_{p'_0} \otimes \dots \otimes d_{m+1} D_{p'_j-m-1} \otimes \dots \otimes D_{p'_n}) \end{aligned} \quad (3.12)$$

From (3.10), it follows that $\sum_{m=0}^{p'_j-1} d_{m+1} D_{p'_j-m-1} = p'_j \cdot D_{p'_j}$ and hence:

$$\sum_{m=0}^{p'_j-1} (D_{p'_0} \otimes \dots \otimes d_{m+1} D_{p'_j-m-1} \otimes \dots \otimes D_{p'_n}) = p'_j \cdot (D_{p'_0} \otimes \dots \otimes D_{p'_j} \otimes \dots \otimes D_{p'_n}) \quad (3.13)$$

Combining (3.11), (3.12) and (3.13), it follows that:

$$\begin{aligned} \Delta^n(D_{K+1}) &= \frac{1}{K+1} \left(\sum_{\sum_{i=0}^n p'_i = K+1} \sum_{j=0, p'_j \geq 1}^n p'_j \cdot (D_{p'_0} \otimes \dots \otimes D_{p'_j} \otimes \dots \otimes D_{p'_n}) \right) \\ &= \frac{1}{K+1} \left(\sum_{\sum_{i=0}^n p'_i = K+1} (K+1) \cdot (D_{p'_0} \otimes \dots \otimes D_{p'_j} \otimes \dots \otimes D_{p'_n}) \right) \\ &= \sum_{\sum_{i=0}^n p'_i = K+1} (D_{p'_0} \otimes \dots \otimes D_{p'_j} \otimes \dots \otimes D_{p'_n}) \end{aligned} \quad (3.14)$$

This proves the result of (3.9) for $K+1$.

□

4 Action on bivariate q -Hochschild cohomology groups

Let A be a commutative algebra over \mathbb{C} and let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let Y be a pointed simplicial finite set. In this section, we will define the bivariate q -Hochschild cohomology groups $\{HH_Y^n(A, A)\}_{n \in \mathbb{Z}}$ of A of order Y and show that a derivation D on A induces a morphism $\underline{L}_D^{Y,n}(A, A) : {}_qHH_Y^n(A, A) \longrightarrow {}_qHH_Y^n(A, A)$. For the ordinary bivariate Hochschild cohomology groups $\{HH^n(A, A)\}_{n \in \mathbb{Z}}$, we have already studied this morphism in [1]. For the definition and properties of ordinary bivariate Hochschild cohomology, we refer the reader to [7, § 5.1] (see also the original paper of Jones and Kassel [4]). We start by defining the bivariate q -Hochschild cohomology groups of order Y .

Definition 4.1. Let $(\mathcal{L}^Y(A), {}_qb)$ be the N -complex corresponding to the simplicial vector space $\mathcal{L}^Y(A)$ as defined in (2.5). We consider the q -Hom complex $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of these N -complexes which is defined as follows:

$$\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n := \prod_{i \in \mathbb{Z}} Hom_{Vect}(\mathcal{L}^Y(A)_i, \mathcal{L}^Y(A)_{i+n}) \quad (4.1)$$

Further, if the family $f = \{f_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+n}\}_{i \in \mathbb{Z}}$ is an element of $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, then the differential ${}_q\partial_n : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{n-1}$ is defined by setting:

$$\begin{aligned} {}_q\partial_n(f) &:= \{{}_q\partial_n(f)_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+n-1}\}_{i \in \mathbb{Z}} \\ {}_q\partial_n(f)_i &= {}_qb_{i+n} \circ f_i - q^n f_{i-1} \circ {}_qb_i \end{aligned} \quad (4.2)$$

For any given $n \in \mathbb{Z}$, we define the bivariate q -Hochschild cohomology group ${}_qHH_Y^n(A, A)$ of A of order Y to be the homology object

$${}_qHH_Y^n(A, A) := H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \quad (4.3)$$

of the N -complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$.

We mention that it follows from [5, Proposition 1.8] that the q -Hom complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$ as defined in (4.1) and (4.2) is also an N -complex. We now make the convention that if $M = \oplus_{i \in \mathbb{Z}} M_i$ is a graded vector space and $f = \{f_i : M_i \longrightarrow M_{i+m}\}_{i \in \mathbb{Z}}$ and $g = \{g_i : M_i \longrightarrow M_{i+n}\}_{i \in \mathbb{Z}}$ are two morphisms of homogenous degree m and n respectively, we will write $[f, g] := f \circ g - q^{mn} g \circ f$ for their graded q -commutator.

Lemma 4.2. Let $L^m = \{L_i^m\}_{i \in \mathbb{Z}}$ denote a collection of maps $L_i^m : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+m}$. Given an element $f = \{f_i\}_{i \in \mathbb{Z}}$ in $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, we define $\underline{L}^m(f) \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{m+n}$ by setting:

$$\underline{L}^m(f)_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+m+n} \quad \underline{L}^m(f)_i := L_{i+n}^m \circ f_i - q^{mn} f_{i+m} \circ L_i^m \quad (4.4)$$

Then, if $q^{2m} = 1$, the endomorphism $\underline{L}^m : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of homogenous degree m satisfies the following relation:

$$[{}_q\partial, \underline{L}^m](f) = [{}_qb, L^m]f + q^{mn+m+n} f[L^m, {}_qb] \quad \forall f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n, n \in \mathbb{Z} \quad (4.5)$$

Proof. We consider:

$$\begin{aligned}
((q\partial \circ \underline{L}^m)(f))_i &= {}_q b_{i+m+n} \circ \underline{L}^m(f)_i - q^{m+n} \underline{L}^m(f)_{i-1} \circ {}_q b_i \\
&= {}_q b_{i+m+n} \circ L_{i+n}^m \circ f_i - q^{mn} {}_q b_{i+m+n} \circ f_{i+m} \circ L_i^m \\
&\quad - q^{m+n} L_{i+n-1}^m \circ f_{i-1} \circ {}_q b_i + q^{mn+m+n} f_{i+m-1} \circ L_{i-1}^m \circ {}_q b_i \\
((\underline{L}^m \circ q\partial)(f))_i &= L_{i+n-1}^m \circ q\partial(f)_i - q^{m(n-1)} q\partial(f)_{i+m} \circ L_i^m \\
&= L_{i+n-1}^m \circ {}_q b_{i+n} \circ f_i - q^n L_{i+n-1}^m \circ f_{i-1} \circ {}_q b_i \\
&\quad - q^{m(n-1)} {}_q b_{i+m+n} \circ f_{i+m} \circ L_i^m + q^{mn-m+n} f_{i+m-1} \circ {}_q b_{i+m} \circ L_i^m
\end{aligned} \tag{4.6}$$

From (4.6), it follows that:

$$\begin{aligned}
([{}_q \partial, \underline{L}^m](f))_i &= ((q\partial \circ \underline{L}^m)(f))_i - q^{-m}((\underline{L}^m \circ q\partial)(f))_i \\
&= ({}_q b_{i+m+n} \circ L_{i+n}^m - q^{-m} L_{i+n-1}^m \circ {}_q b_{i+n}) \circ f_i + f_{i+m-1} \circ q^{mn+m+n} (L_{i-1}^m \circ {}_q b_i - q^{-2m} (q^{-m} {}_q b_{i+m} \circ L_i^m)) \\
&\quad - q^{mn} (1 - q^{-2m}) {}_q b_{i+m+n} \circ f_{i+m} \circ L_i^m - q^{m+n} (1 - q^{-2m}) L_{i+n-1}^m \circ f_{i-1} \circ {}_q b_i
\end{aligned}$$

Combining with the fact that $q^{2m} = 1$, it follows from the above expression that:

$$[{}_q \partial, \underline{L}^m](f) = [{}_q b, L^m]f + q^{mn+m+n} f[L^m, {}_q b] \tag{4.7}$$

□

Proposition 4.3. *Let $q \in \mathbb{C}$ be a primitive N -th root of unity. Let A be a commutative algebra over \mathbb{C} and let $D : A \rightarrow A$ be a derivation on A . Let Y be a pointed simplicial finite set. Then, for each $n \in \mathbb{Z}$, the derivation D on A induces a morphism*

$$\underline{L}_D^{Y,n} : {}_q HH_Y^n(A, A) \rightarrow {}_q HH_Y^n(A, A) \tag{4.8}$$

on the bivariant q -Hochschild cohomology groups of order Y .

Proof. From the proof of Proposition 2.6, we know that the derivation D induces an endomorphism $L_D^Y : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)$ of the simplicial vector space $\mathcal{L}^Y(A)$. Accordingly, we have a collection of maps $L_D^Y = \{L_{D,i}^Y : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_i\}_{i \in \mathbb{Z}}$ determined by the endomorphism L_D^Y . Applying Lemma 4.2 with $m = 0$ (and hence $q^{2m} = 1$), it follows that L_D^Y determines a morphism

$$\underline{L}_D^Y : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \rightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \tag{4.9}$$

of homogeneous degree $m = 0$ satisfying:

$$[{}_q \partial, \underline{L}_D^Y](f) = [{}_q b, L_D^Y]f + q^n f[L_D^Y, {}_q b] \quad \forall f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n, n \in \mathbb{Z} \tag{4.10}$$

Again, since $L_D^Y : \mathcal{L}^Y(A) \rightarrow \mathcal{L}^Y(A)$ is a morphism of simplicial vector spaces, the morphisms $\{L_{D,i}^Y : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_i\}_{i \in \mathbb{Z}}$ commute with the face maps $d_i^j : \mathcal{L}^Y(A)_i \rightarrow \mathcal{L}^Y(A)_{i-1}$, $0 \leq j \leq i$, $i \geq 0$ of the simplicial vector space $\mathcal{L}^Y(A)$. By definition, ${}_q b_i := \sum_{j=0}^i q^j d_i^j$ and hence we have:

$$[{}_q b, L_D^Y] = [L_D^Y, {}_q b] = 0 \tag{4.11}$$

Applying this to (4.10), it follows that:

$$[_q\partial, \underline{L}_D^Y] = {}_q\partial \circ \underline{L}_D^Y - q^{-m} \underline{L}_D^Y \circ {}_q\partial = {}_q\partial \circ \underline{L}_D^Y - \underline{L}_D^Y \circ {}_q\partial = 0 \quad (4.12)$$

From (4.12), it follows that the endomorphism $\underline{L}_D^Y : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \rightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of degree zero commutes with the differential ${}_q\partial$ on the N -complex $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$. This induces morphisms ($\forall n \in \mathbb{Z}$):

$$\begin{aligned} {}_qHH_Y^n(A, A) &= H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \\ &\quad \underline{L}_D^{Y,n} \downarrow \\ {}_qHH_Y^n(A, A) &= H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \end{aligned} \quad (4.13)$$

on the bivariant q -Hochschild cohomology groups of order Y . \square

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